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Spectrum of the Laplacian and Riesz transform on locally symmetric spaces

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Abstract

We assume that the discrete part of the spectrum of the Laplacian on a non-compact locally symmetric space is non-empty and we prove that the Riesz transform is bounded on L^p for all p in an interval around 2.

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1. Introduction and statement of the results

Let M be a complete, non-compact, connected Riemannian manifold. Let us denote by dx the Riemannian measure and by ∇ the gradient. We shall also denote by L^p the space $L^p(M, dx)$, $p \geqslant 1$. If $|.|$ is the length in the tangent space then one can define the (positive) Laplace–Beltrami operator Δ, as well as its square root $\Delta^{1/2}$, as self adjoint and positive operators on L^2 by the formula

$$(\Delta f, f) = \||\nabla f|\|_2^2 = \left\|\Delta^{1/2} f\right\|_2^2, \quad f \in C_0^\infty(M).$$

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Hence the Riesz transform $\nabla\Delta^{-1/2}$ is bounded on L^2. The basic issue to ask, which was raised in [31], is for which complete non-compact Riemannian manifold, and for which $p \in (1,\infty)$, the Riesz transform is bounded on L^p i.e. there exists a constant $c_p > 0$ such that

$$\||\nabla f|\|_p \leqslant c_p \left\|\Delta^{1/2} f\right\|_p, \quad f \in C_0^\infty(M). \tag{1.1}$$

In [9,10] Bakry proves the L^p-boundedness of the Riesz transform when $\mathrm{Ric}(M) \geqslant 0$ and Lohoué [21] on a class of Cartan–Hadamard manifolds which contains non-compact symmetric spaces. In [4] Anker proves the $L^1 - L^1_{\text{weak}}$ continuity on non-compact symmetric spaces. In [6] Auscher, Coulhon, Duong and Hofmann study the Riesz transform in a general Riemannian context which covers the above cases, using estimates of the heat kernel and its gradient.

In the present work we study the Riesz transform on a non-compact locally symmetric space. To state our results we need to recall few basic concepts about symmetric spaces. These standard facts can be found in [17].

Let G be a non-compact and connected semi-simple Lie group with finite center. We denote by K a compact maximal subgroup of G and we consider the symmetric space $X = G/K$.

Let us denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) the Lie algebra of G (resp. K) and let $\mathfrak{p}$ the subspace of $\mathfrak{g}$ which is orthogonal to $\mathfrak{k}$ with respect to the Killing form. We recall that the restriction of the Killing form on $\mathfrak{p}$ is positive and it defines a Riemannian structure on X. We shall denote by Δ the Laplacian and by $d_X(.,.)$ the Riemannian distance.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We denote by $\mathfrak{a}^*$ the dual of $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^*$ we set $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}\colon [H,x] = \alpha(H)$ for every $H \in \mathfrak{a}\}$. If α and $\mathfrak{g}_\alpha$ are non-zeros we say that α is a restricted root. We denote by $m_\alpha = \dim \mathfrak{g}_\alpha$ the multiplicity of the root α.

Let $\Sigma^+ \subset \mathfrak{a}^*$ be a choice of positive roots. A fundamental quantity is

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

Since the Killing form on $\mathfrak{a}$ is positive, it induces an inner product on $\mathfrak{a}^*$, and so $\|\rho\|$ is well defined. It is well known that the spectrum of Δ on $L^2(X)$ is equal to $[\|\rho\|^2, \infty)$.

Let Γ be a discrete and torsion free subgroup of G. We denote by M the locally symmetric space $\Gamma\backslash G/K$. Since Γ is torsion free, M equipped with the projection of the Riemannian structure of X, becomes a complete Riemannian manifold with negative Ricci curvature. We shall denote also by Δ the Laplacian, by $d(.,.)$ and dx the Riemannian distance and measure of M.

We recall that the L^2-spectrum of Δ on a non-compact locally symmetric space is in general unknown. In the present work we shall assume that it is equal to

$$\{\lambda_0, \ldots, \lambda_m\} \cup \left[\|\rho\|^2, \infty\right), \tag{1.2}$$

where the eigenvalues $0 \leqslant \lambda_0 < \cdots < \lambda_m$ are of finite multiplicity. This is the case if M is the quotient of the hyperbolic space $\mathbb{H}^{n+1}$ by a geometrically finite Kleinian group Γ, i.e. when $M = \Gamma\backslash\mathbb{H}^{n+1} = \Gamma\backslash SO(n+1,1)/SO(n+1)$ [18]. Note that in this case $\|\rho\| = n/2$.

In order to estimate the bottom of the spectrum λ_0 we need the critical exponent $\delta(\Gamma)$ of the group Γ which is defined as follows. Let x_0 be a fixed point of X and for $R > 0$ we denote by n_R the cardinal of the set $\{\gamma \in \Gamma\colon d_X(x_0, \gamma x_0) < R\}$. Then

$$\delta(\Gamma) = \limsup_{R\to\infty} \frac{\log n_R}{R}.$$

We always have that $0 \leqslant \delta(\Gamma) \leqslant 2\|\rho\|$ [32, p. 33].

Let $\mathfrak{a}^+$ be a positive Weyl chamber associated with a choice of a set of positive roots $\Sigma^+ \subset \mathfrak{a}^*$. We set

$$\rho_{\min} = \min_{H \in \overline{\mathfrak{a}^+}, \|H\|=1} \rho(H).$$

In [32, Theorem 3.8] Weber, following Leuzinger [19], proved that the point spectrum of Δ is empty when $0 \leqslant \delta(\Gamma) \leqslant \rho_{\min}$. So, the point spectrum appears in the case when $\delta(\Gamma) > \rho_{\min}$. Further, if $\delta(\Gamma) < \|\rho\| + \rho_{\min}$, then $\lambda_0 > 0$, while if $\delta(\Gamma) \geqslant \|\rho\| + \rho_{\min}$ then λ_0 may be equal to 0. This is for example the case when $\mathrm{vol}(M) < \infty$, since the constants belong in L^2.

Note that in the case of a Kleinian group $M = \Gamma \backslash \mathbb{H}^{n+1}$, if $\delta(\Gamma) > n/2$, then $\lambda_0 = \delta(\Gamma)(n - \delta(\Gamma))$. Thus $\lambda_0 = 0$ if $\delta(\Gamma) = n$.

Before stating our result on the L^p-boundedness of the Riesz transform, let us make clear that its proof depends on the properties of the L^2-eigenfunctions associated to the point spectrum. In fact we shall show in Theorem 1 below that they belong also to $L^p(M)$ for p in some interval (r_1, r_2) around 2. This fact is a generalization of some results in [14] (see also [30]) and it is inspired from the work [24] of N. Lohoué.

Theorem 1. *If $\delta(\Gamma) > \rho_{\min}$, then every L^2-eigenfunction u_j with eigenvalue λ_j, $j \leqslant m$, belongs to L^p for all $p \in (r_1, r_2)$, where*

$$r_1 = 2\left\{\left(1 - \left(\lambda_m/\|\rho\|^2\right)\right)^{1/2} + 1\right\}^{-1},$$

and

$$r_2 = 2\left\{\left(1 - \left(\lambda_m/\|\rho\|^2\right)\right)^{1/2} + 1\right\}.$$

Let r_2' be the conjugate of r_2. Note that $r_1 \leqslant r_2'$.

Theorem 2.

(i) *If $0 \leqslant \delta(\Gamma) \leqslant \rho_{\min}$, then for all $p \in (1, \infty)$, there is a constant $c_p > 0$ such that*

$$\||\nabla f|\|_p \leqslant c_p \left\|\Delta^{1/2} f\right\|_p, \quad f \in C_0^\infty(M). \tag{1.3}$$

(ii) *If $\delta(\Gamma) > \rho_{\min}$ and $\lambda_0 \neq 0$, then* (1.3) *is valid for all $p \in (r_2', r_2)$.*

(iii) *If $\delta(\Gamma) \geqslant \|\rho\| + \rho_{\min}$ and $\lambda_0 = 0$, then* (1.3) *is valid for all $p \in (r_2', r_2)$ and for all $f \in C_0^\infty(M)$ such that*

$$\int_M u_0^j(x) f(x)\, dx = 0,$$

where u_0^j, $j \leqslant k_0$, are the L^2-harmonic functions.

For the proof of Theorem 2 we use Theorem 1 and the following local version of (1.3): for all $p \in (1, \infty)$, there are positive constants c_1 and c_2 depending on p such that

$$\||\nabla f|\|_p \leqslant c_1 \left\|\Delta^{1/2} f\right\|_p + c_2 \|f\|_p, \quad f \in C_0^\infty(M). \tag{1.4}$$

Inequality (1.4) has been proved by Lohoué in [21] for complete manifolds with bounded geometry and extended by Bakry [10, Theorem 4.1, p. 160], in the case when the injectivity radius is not bounded below.

Next we deal with the case when $M = \Gamma\backslash\mathbb{H}^{n+1}$. In this case the point spectrum is non-empty if $\delta(\Gamma) > n/2$. Using the results of [14], one can replace in Theorem 2 the interval (r_2', r_2) by (r_1, r_1') which is bigger.

Theorem 3. *If $M = \Gamma\backslash\mathbb{H}^{n+1}$, where Γ is geometrically finite Kleinian group with $\delta(\Gamma) > n/2$, then claims* (ii) *and* (iii) *of Theorem* 2 *are valid for all $p \in (r_1, r_1')$.*

The L^p-boundedness of the Riesz transform is extensively studied in various geometric settings, as Riemannian manifolds of polynomial volume growth [5–12,20,27,28], or exponential volume growth [6,10,21], Lie groups [1,22,23], Cartan–Hadamard manifolds and symmetric spaces of non-compact type [4,21], discrete groups [2,16] or graphs [29]. For an extended list of references see [6]. See also [25,26] for the related problem of multipliers on locally symmetric spaces and Kleinian groups.

Finally, let us say a few words about claims (ii) and (iii) of Theorem 2 where it is proved that the Riesz transform is bounded on L^p not for all p, but only for p in an interval (r_2', r_2) around 2. This is due to the fact, we proved in Theorem 1 above, that the L^2-eigenfunctions associated to the discrete spectrum belong in L^p only for $p \in (r_2', r_2)$. The phenomenon where the Riesz transform is bounded on L^p for certain values of p has been observed first by H.-Q. Li [20] and Coulhon and Duong [11]. In [20, Theorem 1] Li proves that on a certain class of conical manifolds, the Riesz transform is bounded on L^p if and only if $p \in (1, p_0)$ for some $p_0 > 2$. Later in [6], it is proved that on manifolds satisfying the doubling volume property and the Poincaré inequality, the Riesz transform is bounded on L^p for $p \in (2, p_0)$ for some $p_0 > 2$, if and only if the heat operator $e^{-t\Delta}$ satisfies

$$\sup_{t>0} \sqrt{t}\,\big\|\nabla e^{-t\Delta}\big\|_{p\to p} < \infty, \tag{1.5}$$

for all p in the same region. As it is observed in [13] the class of conical manifolds treated by Li satisfies the doubling volume property and the Poincaré inequality, and further, (1.5) is satisfied precisely in the region where Li proves the L^p-boundedness of the Riesz transform. Finally, it is worth mentioning that in the case of non-compact symmetric spaces we treat here, the approach of [6] does not apply, since these manifolds do not satisfy the doubling volume property.

Throughout this article the different constants will always be denoted by the same letter c. When their dependence or independence is significant, it will be clearly stated.

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2. Proof of Theorem 1

Let T be an operator on $L^2(M)$ and let us denote by $\mathrm{sp}(T)$ its spectrum. Let us also recall that we made the assumption that

$$\mathrm{sp}(\Delta) = \{\lambda_0, \dots, \lambda_m\} \cup \big[\|\rho\|^2, \infty\big), \tag{2.1}$$

and that the eigenvalues $\lambda_0, \dots, \lambda_m$ are of finite multiplicity. Thus the eigenspace

$$E_j = \big\{f \in L^2(M)\colon\ \Delta f = \lambda_j f\big\}, \quad j \leqslant m,$$

has finite dimension. So, E_j is spanned by a finite number of eigenfunctions u_j^k, $k \leqslant k_j$. For simplicity we write u_j instead of u_j^k.

Let us recall that a point $\tilde{x} \in M$ is identified with the trajectory $\{\gamma x; \gamma \in \Gamma\}$ of the point $x \in X$. Let us denote by $P_t = e^{-t\Delta}$, $t > 0$, the heat semigroup on M and by $p_t(\tilde{x}, \tilde{y})$ its kernel, i.e. the heat kernel on M.

For the proof of Theorem 1 we need the following lemma.

Lemma 1. *Let x_0 be a fixed point in X. If $p > 2$, then for*

$$\beta > \|\rho\|(p-2), \tag{2.2}$$

there exists $c > 0$ such that

$$p_1(\tilde{x}, \tilde{x})^{(p-2)/2} \leqslant c e^{\beta d(\tilde{x}, \tilde{x}_0)}, \quad \text{for all } \tilde{x} \in M. \tag{2.3}$$

Proof. Let

$$P(s; x, y) = \sum_{\gamma \in \Gamma} e^{-s d_X(x,y)}, \quad s > \delta(\Gamma),\ x, y \in X,$$

be the Poincaré series of Γ. As it is shown in [32, p. 46], for $\eta > 0$, there exists a positive constant $c(\eta)$ such that

$$p_1(\tilde{x}, \tilde{x}) \leqslant c(\eta) P\big(\delta(\Gamma) + \eta; x, x\big), \quad x \in X. \tag{2.4}$$

Further, in [32, p. 36], it is proved that for $s > 2\|\rho\|$ and $x_0 \in X$ fixed, there exists a positive constant $c(s, x_0)$ such that

$$P(s; x, x) \leqslant c(s, x_0) e^{s d(\tilde{x}, \tilde{x}_0)}, \quad \text{for all } x \in X. \tag{2.5}$$

Let us now choose η in (2.4) such that

$$\delta(\Gamma) + \eta = 2\big(\|\rho\| + \varepsilon\big), \quad \varepsilon > 0.$$

Using (2.5) we get

$$\begin{aligned} p_1(\tilde{x}, \tilde{x})^{(p-2)/2} &\leqslant c e^{(\|\rho\| + \varepsilon) d(\tilde{x}, \tilde{x}_0)(p-2)} \\ &\leqslant c e^{\beta d(\tilde{x}, \tilde{x}_0)}, \end{aligned}$$

provided that $\beta > \|\rho\|(p-2)$. $\square$

Proof of Theorem 1. Let u_j be an L^2-eigenfunction with eigenvalue λ_j. We have to show that $\|u_j\|_p < \infty$ for all $p \in (r_1, r_2)$. Let us treat first the case $p > 2$. We have

$$P_t u_j(\tilde{x}) = e^{-\lambda_j t} u_j(\tilde{x}) = \int_M p_t(\tilde{x}, \tilde{y}) u_j(\tilde{y})\, d\tilde{y}.$$

Taking $t = 1/2$, and using the semigroup property of $p_t(\tilde{x}, \tilde{y})$, it follows that

$$\begin{aligned} \big|u_j(\tilde{x})\big| &\leqslant e^{\lambda_j/2} \int_M p_{1/2}(\tilde{x}, \tilde{y}) \big|u_j(\tilde{y})\big|\, d\tilde{y} \\ &\leqslant e^{\lambda_j/2} \Big(\int_M p_{1/2}(\tilde{x}, \tilde{y})^2\, d\tilde{y}\Big)^{1/2} \Big(\int_M \big|u_j(\tilde{y})\big|^2\, d\tilde{y}\Big)^{1/2} \\ &= e^{\lambda_j/2} p_1(\tilde{x}, \tilde{x})^{1/2} \|u_j\|_2. \end{aligned} \tag{2.6}$$

We write

$$\begin{aligned}\|u_j\|_p^p &= \int_M \left|u_j(\tilde{x})\right|^{p-2}\left|u_j(\tilde{x})\right|^2 d\tilde{x}\\ &= \int_M \left|u_j(\tilde{x})\right|^{p-2} e^{-\beta d(\tilde{x},\tilde{x}_0)} e^{\beta d(\tilde{x},\tilde{x}_0)}\left|u_j(\tilde{x})\right|^2 d\tilde{x},\end{aligned} \tag{2.7}$$

where $\tilde{x}_0$ is a fixed point in M and β is given by (2.2). Combining (2.6) and (2.7) we get that

$$\|u_j\|_p^p \leqslant e^{(p-2)\lambda_j/2}\|u_j\|_2^{(p-2)} \int_M p_1(\tilde{x},\tilde{x})^{(p-2)/2} e^{-\beta d(\tilde{x},\tilde{x}_0)} e^{\beta d(\tilde{x},\tilde{x}_0)}\left|u_j(\tilde{x})\right|^2 d\tilde{x}. \tag{2.8}$$

Using (2.3) it follows that if β is as above, then

$$\|u_j\|_p^p \leqslant c e^{(p-2)\lambda_j/2}\|u_j\|_2^{(p-2)} \int_M e^{\beta d(\tilde{x},\tilde{x}_0)}\left|u_j(\tilde{x})\right|^2 d\tilde{x}. \tag{2.9}$$

Combining (2.9) with Agmon's L^2-weighted estimate [3, p. 55]

$$\int_M \left|u_j(\tilde{x})\right|^2 e^{2(1-\varepsilon)d(\tilde{x},\tilde{x}_0)(\|\rho\|^2-\lambda_j)^{1/2}}\, d\tilde{x} \leqslant c, \quad \text{for all } \varepsilon > 0, \tag{2.10}$$

it follows that $\|u_j\|_p < \infty$ provided that

$$2(1-\varepsilon)\left(\|\rho\|^2-\lambda_j\right)^{1/2} \geqslant \beta > \|\rho\|(p-2),$$

i.e. when

$$p < 2\left(1-\frac{\lambda_j}{\|\rho\|^2}\right)^{1/2} + 2.$$

The case $p \in (r_1, 2)$ is a particular case of a more general result of M. Taylor [30, pp. 783–784]. So, we shall only give the tools we need for its proof. First, let us recall that M has exponential volume growth; for $R > 0$, the volume $V(x, R)$ of the ball $B(x, R)$ satisfies

$$V(x,R) \sim R^{\frac{a-1}{2}} e^{2\|\rho\|R}, \tag{2.11}$$

where $a = \operatorname{rank} X = \dim \mathfrak{a}$ [32, p. 33].

Using (2.11) and the fact that eigenfunctions u_j with eigenvalue λ_j have exponential decay

$$\int_{B(x_0,R)^c} \left|u_j(x)\right|^2 dx \leqslant c e^{-2R(\|\rho\|^2-\lambda_j)^{1/2}},$$

Taylor proves in [30, pp. 783–784], that $u_j \in L^p$ for all $p \in (r_1, 2)$. $\square$

Let us now present the L^p-properties of the eigenfunctions u_j in some interesting particular cases.

Proposition 1.

(i) *If* $\operatorname{vol}(M) < \infty$*, then* $u_j \in L^p(M)$ *for all* $p \in [1, 2]$.

(ii) *If* $\dim M \geqslant 3$ *and* M *has bounded geometry, then every* L^2*-eigenfunction belongs in* L^p *for all* $p > 2$.

Proof. (i) Just note that for all $p \in [1,2)$, we have that $L^2(M) \subset L^p(M)$.

(ii) In [32, Theorem 3], Weber proved that in this case every L^2-eigenfunction u_j is bounded. So, for $p > 2$, we have that

$$\int_M \left|u_j(x)\right|^p dx \leqslant \|u_j\|_\infty^{p-2}\|u_j\|_2^2 < \infty. \qquad \square$$

Remark 1. In the particular case of Kleinian groups $M = \Gamma\backslash\mathbb{H}^{n+1}$, Davies, Simon and Taylor have obtained in [14] the following results.

(i) If $\mathrm{vol}(M) = \infty$, then every L^2-eigenfunction belongs in L^p for all $p \in (r_1, r_2)$, where

$$r_1 = 2\left\{1 + \left(1 - \left(\lambda_m/\|\rho\|^2\right)\right)^{1/2}\right\}^{-1} \quad \text{and} \quad r_2 = r_1',$$

cf. [14, Propositions 12 and 18]. Note that

$$r_1' = 2\left\{1 - \left(1 - \left(\lambda_m/\|\rho\|^2\right)\right)^{1/2}\right\}^{-1}, \tag{2.12}$$

and that $\|\rho\| = n/2$.

(ii) If $\mathrm{vol}(M) < \infty$, then $r_1 = 1$ and

(iii) If $\dim(M) \geqslant 3$, and M has bounded geometry, then $r_2 = \infty$.

Remark 2. In the case when $M = \Gamma\backslash\mathbb{H}^{n+1}$ and M contains a cusp of rank r, then [15, Theorem 5.4]

$$p_1(\tilde{x}, \tilde{x}) \leqslant ce^{-\delta(\Gamma)(n-\delta(\Gamma))}e^{rd(x,x_0)}.$$

Bearing in mind that $\|\rho\| = n/2$ and arguing as in the proof of Theorem 1, we get that $u_j \in L^p$, provided that

$$p < \frac{4}{r}\left(\frac{n^2}{4} - \lambda_j\right)^{1/2} + 2.$$

3. Proof of Theorems 2 and 3

As it was shown in Theorem 1, all L^2-eigenfunctions u_j, $j = 0, 1, \ldots, m$, belong also to L^p, $p \in (r_2', r_2)$. Let us denote by L_m^p the span in L^p of u_j, $j = 0, 1, \ldots, m$. Since, by our assumption, all the eigenvalues have finite multiplicity, L_m^p is finite dimensional. It follows that

$$L^p = L_m^p \oplus \left(L_m^p\right)^\perp,$$

where

$$\left(L_m^p\right)^\perp = \left\{f \in L^p\colon \langle f, u_j\rangle = 0,\ 0 \leqslant j \leqslant m\right\}$$

is the complement of L_m^p in L^p.

Let us denote by π_m the projection of L^p on L_m^p:

$$\pi_m(f) = \sum_{0\leqslant j\leqslant m} \langle f, u_j\rangle u_j, \quad f \in L^p.$$

An operator T on L^p is then written as

$$Tf = T\pi_m(f) + T(I - \pi_m)(f).$$

We shall use the above decomposition to prove the following lemma.

Lemma 2. *If $\lambda_0 \neq 0$, then $\Delta^{-1/2}$ is bounded on L^p for all $p \in (r_2', r_2)$.*

Proof. We shall first show that $\Delta^{-1/2}\pi_m$ is bounded on L^p. Since $\lambda_j \neq 0$ for all $j \leqslant m$, we have that

$$\begin{aligned}
\Delta^{-1/2}\pi_m(f) &= \Delta^{-1/2}\left(\sum_{0\leqslant j\leqslant m} \langle f, u_j\rangle u_j\right) \\
&= \sum_{0\leqslant j\leqslant m} \langle f, u_j\rangle \Delta^{-1/2} u_j \\
&= \sum_{0\leqslant j\leqslant m} \langle f, u_j\rangle \lambda_j^{-1/2} u_j. \qquad (3.1)
\end{aligned}$$

By Theorem 1, $u_j \in L^p$ for all $p \in (r_2, r_2')$. Thus if $f \in L^p$ and q is the conjugate of p, by (3.1) we get that

$$\begin{aligned}
\left\|\Delta^{-1/2}\pi_m(f)\right\|_p &\leqslant \sum_{0\leqslant j\leqslant m} \left|\langle f, u_j\rangle\right| \lambda_j^{-1/2} \|u_j\|_p \\
&\leqslant \sum_{0\leqslant j\leqslant m} \lambda_j^{-1/2} \|f\|_p \|u_j\|_q \|u_j\|_p \\
&\leqslant c\|f\|_p. \qquad (3.2)
\end{aligned}$$

It remains to show that $\Delta^{-1/2}(I - \pi_m)$ is also bounded on L^p. For that we use the following Laplace transform formula:

$$\begin{aligned}
\Delta^{-1/2}(I - \pi_m)f &= c\int_0^\infty e^{-t\Delta}(I - \pi_m)f \frac{dt}{\sqrt{t}} \\
&= c\int_0^\infty P_t(I - \pi_m)f \frac{dt}{\sqrt{t}}. \qquad (3.3)
\end{aligned}$$

Next, we shall prove that there is a positive constant $c(p)$ such that

$$\left\|P_t(I - \pi_m)\right\|_{p\to p} \leqslant e^{-tc(p)}, \qquad (3.4)$$

for all $p \in (r_2', r_2)$.

Combining (3.3) and (3.4) we obtain that

$$\begin{aligned}
\left\|\Delta^{-1/2}(I - \pi_m)f\right\|_p &\leqslant c\int_0^\infty \left\|P_t(I - \pi_m)f\right\|_p \frac{dt}{\sqrt{t}} \\
&\leqslant c\int_0^\infty e^{-tc(p)}\|f\|_p \frac{dt}{\sqrt{t}} \leqslant c\|f\|_p. \qquad (3.5)
\end{aligned}$$

Thus, to complete the proof of the lemma, it remains to prove (3.4). For that we write

$$P_t = P_t\pi_m + P_t(I - \pi_m).$$

It is easy to see that P_t leaves invariant both L_m^p and $(L_m^p)^\perp$. This implies that $P_t\pi_m$ is an operator on L_m^p and $P_t(I - \pi_m)$ on $(L_m^p)^\perp$. Clearly, the L^2-spectrum of $P_t\pi_m$ is equal to $\{e^{-t\lambda_0}, \ldots, e^{-t\lambda_m}\}$. This, combined with the fact the spectrum of P_t is equal to

$$\{e^{-t\lambda_0}, \ldots, e^{-t\lambda_m}\} \cup [e^{-t\|\rho\|^2}, \infty),$$

implies that the L^2-spectrum of $P_t(I - \pi_m)$ is equal to $[e^{-t\|\rho\|^2}, \infty)$.

This gives that

$$\| P_t(I - \pi_m)\|_{2\to 2} \leqslant e^{-t\|\rho\|^2}.$$

Also, since P_t is a contraction on L^p for all $p \geqslant 1$, it follows that

$$\begin{aligned} \| P_t(I - \pi_m) f\|_{r_2} &\leqslant \|P_t\|_{r_2\to r_2} \|(I - \pi_m) f\|_{r_2} \\ &\leqslant \|(I - \pi_m) f\|_{r_2} \\ &\leqslant c\|f\|_{r_2}. \end{aligned}$$

By interpolation and duality we have that

$$\| P_t(I - \pi_m)\|_{p\to p} \leqslant e^{-tc(p)},$$

for all $p \in (r_2', r_2)$ and the proof of (3.4) is complete. □

Lemma 3. *If $\lambda_0 = 0$, then $\Delta^{-1/2}$ is bounded on $(L_0^p)^\perp$ for all $p \in (r_2, r_2')$.*

Proof. Let us denote by u_0^j, $j \leqslant k$, the L^2-eigenfunctions with eigenvalue 0, i.e. the L^2-harmonic functions. Let L_0^p be the span in L^p of u_0^j, $j \leqslant k$. If for example $\mathrm{vol}(M) < \infty$, then $1 \in L_0^p$ for all $p \geqslant 1$.

Let us now assume that for some $j \leqslant k$, u_0^j satisfies

$$\|u_0^j\|_2 \leqslant c\|\Delta^{1/2} u_0^j\|_2 = c\|\lambda_0^{1/2} u_0^j\|_2 = 0.$$

It follows that $u_0^j = 0$, and consequently $\Delta^{-1/2}$ is not bounded on L_0^p.

Since $\dim L_0^p < \infty$, we have that

$$L^p = L_0^p \oplus (L_0^p)^\perp$$

where

$$(L_0^p)^\perp = \{f \in L^p\colon \langle f, u_0^j\rangle = 0,\ j \leqslant k\}.$$

Proceeding as in the previous case when $\lambda_0 \neq 0$, one can see that the L^2-spectrum of P_t on $(L_0^p)^\perp$ is equal to

$$\{e^{-t\lambda_1}, \ldots, e^{-t\lambda_m}\} \cup [e^{-t\|\rho\|^2}, \infty).$$

But $\lambda_1 \neq 0$ and the same arguments as in Lemma 2, allow us to prove that $\Delta^{-1/2}$ is bounded on $(L_0^p)^\perp$. □

End of proof of Theorems 2 and 3. To prove claim (i) of Theorem 2 we recall that in this case the L^2-spectrum of Δ is equal to $[\|\rho\|^2, \infty)$. This implies that

$$\|P_t\|_{2\to 2} \leqslant e^{-t\|\rho\|^2}.$$

Also, $\|P_t\|_{1\to 1} \leqslant 1$ and by interpolation and duality, we have that

$$\|P_t\|_{p\to p} \leqslant e^{-tc(p)},$$

for all $p \in (1, \infty)$. Arguing as in (3.5) we get that $\|\Delta^{-1/2}\|_{p\to p} \leqslant c_p < \infty$ for all $p \in (1, \infty)$.

Claims (ii) and (iii) of Theorem 2 follow from Lemma 2 and Lemma 3 while for the proof of Theorem 3, instead of Theorem 1 we use the results of [14] we presented in the Remark 1. □

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